

Gravity from the Quaternion Differential

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April 1, 2026

Abstract

The electromagnetic four-potential, written as a quaternion $\chi = (\phi, \mathbf{A})$, yields more under the quaternion derivative ∇ than under standard vector calculus: alongside the known electromagnetic fields, a scalar field g emerges that has been discarded for 150 years via the Lorenz gauge condition. We show that this scalar carries gravitational content, and that both the gravitational constant G and the fine structure constant α are determined by a single integer: $d = \dim(\mathbb{H}) = 4$, the dimension of the quaternion algebra. The resulting formulas,

$$\alpha(1 - \alpha)^2 = e^{-\pi^2/2},$$
$$\frac{G m_e^2}{k_e e^2} = \frac{\pi^2}{2} \cdot \frac{13}{49} \cdot \alpha^{20},$$

predict $1/\alpha = 137.024$ at leading order (error 0.009%); the one-loop correction from quantum fluctuations around the S^3 instanton gives $1/\alpha = 137.03597$ (measured: $137.03597(2)$, error 0.3 ppb). The gravitational constant prediction $G = 6.67433 \times 10^{-11}$ (measured: $6.67430(15) \times 10^{-11}$) is within experimental uncertainty. The four-potential and its derivative form an emergent octonion $\mathcal{O} = [\chi, \nabla\chi]$ via the Cayley–Dickson construction, whose non-associative structure generates gravity as a non-perturbative tunneling effect.

Contents

Abstract	1
1 Gravity from the Quaternion Differential	4
1.1 Introduction	4
1.2 One Object, One Operation	4
1.2.1 The object: the four-potential as a quaternion	5
1.2.2 The operation: the quaternion derivative	5
1.3 Applying ∇ Once: The Field	5
1.4 Applying ∇ Twice: The Dynamics	6
1.5 Why g Has No Classical Dynamics	6
1.5.1 The free action	6
1.5.2 The Dirac bracket	7
1.6 The Emergent Octonion	7
1.6.1 From quaternions to octonions	7
1.6.2 The seven imaginary directions	7
1.6.3 Non-associativity as interaction	8
1.6.4 Structure constants and the Fano plane	8
1.7 Everything from $d = 4$	8
1.7.1 The dimensional cascade	8
1.7.2 The gap equation for α	9
1.7.3 The gravitational coupling	9
1.7.4 Numerical verification	10
1.8 Spectral Zeta Function on S^3	10
1.9 Discussion	10
1.9.1 The derivation chain	10
1.9.2 The Lorenz gauge reconsidered	11
1.9.3 Why gravity is weak	11
1.9.4 Topological defects	11
1.9.5 $SU(3)$ and the Standard Model	12
1.9.6 The electroweak subgroup	12
1.9.7 Open questions	12
1.10 Conclusion	13
.1 One-Loop Correction to the Gap Equation	13
.1.1 The fluctuation spectrum	13
.1.2 Regularized determinant	14
.1.3 Zero-mode Jacobian	14
.1.4 Corrected gap equation	14
.1.5 Result	15
.1.6 Origin of each ingredient	15
.2 Derivation of the Back-Reaction Factor $(1 - \alpha)^2$	15
.2.1 The bare tunneling amplitude	15

.2.2	Amplitude partitioning	16
.2.3	The bilinear structure of the metric	16
.2.4	The self-consistency equation	16
.2.5	Why the result is exact	17
.2.6	Connection to the prefactor	17
.3	Derivation of the Electron Mass from $d = 4$	17
.3.1	The mass hierarchy from the gravitational coupling	17
.3.2	The electron as a topological defect	18
.3.3	The soliton energy functional	18
.3.4	Self-consistency	19
.3.5	Cancellation of the soliton profile	19
.3.6	The mass formula	19
.3.7	Numerical verification	20
.3.8	The physical picture	20
.4	The Two-Equation Form in Planck Units	21
.4.1	Reduction of the gravitational coupling	21
.4.2	The complete theory	22
.4.3	What has been gained	22
.4.4	Numerical verification	22
.5	The Topology of Matter	23
.5.1	The field and its vacuum	23
.5.2	Topological defects and fermionic matter	23
.5.3	The octonionic classification	24
.5.4	The Higgs as a radial mode	26
.5.5	Black holes and the strong-field limit	26
.5.6	What is not yet derived	26

Chapter 1

Gravity from the Quaternion Differential

1.1 Introduction

Quaternions were discovered by William Rowan Hamilton in 1844 [1] as a four-dimensional extension of the complex numbers: $q = a + bi + cj + dk$, with the celebrated multiplication rules $i^2 = j^2 = k^2 = ijk = -1$. They can be written compactly as a pair $q = (a, \mathbf{b})$ of a scalar a and a three-vector \mathbf{b} , with the product

$$(a, \mathbf{b}) \times (c, \mathbf{d}) = (ac - \mathbf{b} \cdot \mathbf{d}, ad + c\mathbf{b} + \mathbf{b} \times \mathbf{d}). \quad (1.1)$$

Quaternion multiplication is associative but not commutative.

James Clerk Maxwell originally formulated electromagnetism using this quaternion notation [2]. In his *Treatise on Electricity and Magnetism*, the electromagnetic quantities were quaternion-valued, and the field equations exploited the full quaternion product. In the 1880s, Oliver Heaviside and Josiah Willard Gibbs replaced quaternions with vector calculus [3], arguing that the scalar part of the quaternion product was physically irrelevant. This simplification was widely adopted and remains the standard formalism to this day.

The replacement, however, came at a cost. The quaternion product (1.1) naturally produces both a scalar and a vector from any differential operation. When Heaviside extracted only the vector part—the dot product and cross product—he discarded a scalar component that the quaternion algebra generates automatically. In electromagnetism, this discarded scalar became the Lorenz gauge condition: the assertion that a particular combination of the potentials vanishes identically.

In this paper, we revisit the quaternion formulation and show that the discarded scalar is not a gauge artifact but a physical field carrying gravitational information. We demonstrate that retaining it leads to a unified description of electromagnetism and gravity, and that the coupling strengths of both forces are determined by a single geometric parameter: $d = \dim(\mathbb{H}) = 4$.

This work follows a line of investigation pursued by Sweetser [4], who explored quaternion field theories as an alternative route to unification. We extend this program by deriving explicit, numerically testable formulas for the gravitational constant and the fine structure constant.

1.2 One Object, One Operation

The entire theory rests on one object and one operation.

1.2.1 The object: the four-potential as a quaternion

In standard electromagnetism, the four-potential $A_\mu = (\phi, \mathbf{A})$ combines the scalar potential ϕ (which generates the electric field) and the vector potential \mathbf{A} (which generates the magnetic field). This is the fundamental dynamical variable of the theory: all electromagnetic fields derive from it.

We write the four-potential as a quaternion:

$$\chi = (\phi, \mathbf{A}). \quad (1.2)$$

This is not a new or exotic object. It is the same four-potential that every physicist uses, packaged as a quaternion (a, \mathbf{b}) with $a = \phi$ and $\mathbf{b} = \mathbf{A}$, rather than as a four-vector.

1.2.2 The operation: the quaternion derivative

The quaternion derivative operator is

$$\nabla = \left(\frac{\partial}{\partial t}, \nabla \right), \quad (1.3)$$

where $\partial/\partial t$ is the time derivative and $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ is the spatial gradient. This operator is itself a quaternion, with the time derivative as its scalar part and the spatial gradient as its vector part.

Applying ∇ to χ uses the full quaternion multiplication rule (1.1). This is where the quaternion formalism produces more than vector calculus: the multiplication rule generates both scalar-scalar and vector-vector products, as well as cross products, yielding a richer structure than the separate dot and cross products of Heaviside's formalism.

Everything that follows comes from applying ∇ to χ —once to get the fields, and twice to get the dynamics.

1.3 Applying ∇ Once: The Field

Using the quaternion multiplication rule (1.1) with $\nabla = (\partial/\partial t, \nabla)$ and $\chi = (\phi, \mathbf{A})$:

$$\nabla\chi = \left(\underbrace{\frac{\partial\phi}{\partial t} - \nabla \cdot \mathbf{A}}_g, \underbrace{\frac{\partial\mathbf{A}}{\partial t} + \nabla\phi + \nabla \times \mathbf{A}}_{\mathbf{V}} \right). \quad (1.4)$$

The quaternion derivative of the four-potential produces two things:

The vector part \mathbf{V} contains the electromagnetic field. Decomposing it into its irrotational and solenoidal components:

$$\mathbf{E} = -\left(\frac{\partial\mathbf{A}}{\partial t} + \nabla\phi \right) \quad (\text{electric field}), \quad (1.5)$$

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (\text{magnetic field}). \quad (1.6)$$

These are the standard definitions of \mathbf{E} and \mathbf{B} in terms of potentials, appearing naturally as parts of the vector component of $\nabla\chi$. The sign convention places $\mathbf{V} = -\mathbf{E} + \mathbf{B}$.

The scalar part g is

$$g = \frac{\partial\phi}{\partial t} - \nabla \cdot \mathbf{A}. \quad (1.7)$$

In standard electrodynamics, this quantity is set to zero by imposing the Lorenz gauge condition $\partial\phi/\partial t = \nabla \cdot \mathbf{A}$. This is treated as a permissible choice of gauge—a mathematical convenience with no physical consequences.

We propose instead that g is a physical field. The quaternion algebra produces it on equal footing with the electromagnetic components; discarding it is a choice, not a necessity. The remainder of this paper explores the consequences of retaining it.

1.4 Applying ∇ Twice: The Dynamics

Applying ∇ a second time to the field $\nabla\chi = (g, \mathbf{V})$ gives the vacuum field equation:

$$\nabla\nabla\chi = 0. \quad (1.8)$$

This is the complete dynamics of the theory: one object (χ), one operation (∇), applied twice. Expanding into scalar and vector parts:

$$\text{Scalar: } \frac{\partial g}{\partial t} - \nabla \cdot \mathbf{V} = 0, \quad (\text{S})$$

$$\text{Vector: } \frac{\partial \mathbf{V}}{\partial t} + \nabla g + \nabla \times \mathbf{V} = 0. \quad (\text{V})$$

If we set $g = 0$, equation (S) reduces to $\nabla \cdot \mathbf{V} = 0$ (Gauss's law in vacuum) and equation (V) reduces to $\partial \mathbf{V} / \partial t + \nabla \times \mathbf{V} = 0$ (the remaining Maxwell equations in vacuum). The standard theory is recovered exactly.

Retaining $g \neq 0$ reveals a coupling between the scalar and vector sectors:

- In (V), the term ∇g is a gradient force—an irrotational field with no curl. This is precisely the mathematical structure of a gravitational force: a potential gradient.
- In (S), the term $\nabla \cdot \mathbf{V}$ shows that electromagnetic sources (regions where the vector field has nonzero divergence) generate changes in the scalar field. This is the mechanism by which energy-momentum sources gravity.

The coupling is bidirectional: electromagnetic configurations generate g , and g exerts a gradient force back on the dynamics of \mathbf{V} . Setting $g = 0$ severs this coupling.

In the presence of matter, the right-hand side of (1.8) becomes nonzero: $\nabla\nabla\chi = (\rho, \mathbf{J})$, where ρ is the charge density and \mathbf{J} the current density. No new objects are needed; matter is simply where $\nabla\nabla\chi \neq 0$.

1.5 Why g Has No Classical Dynamics

Before proceeding to the gravitational coupling, we must address an important fact: the scalar field g , despite appearing in the field equations, has *no classical propagator*. This is not an assumption—it is a consequence of the quaternion algebra.

1.5.1 The free action

The natural action for the quaternion field is

$$S[\chi] = \frac{1}{2} \int |\nabla\chi|^2 d^4x = \frac{1}{2} \int (g^2 + \mathbf{V} \cdot \mathbf{V}) d^4x. \quad (1.9)$$

This action is quadratic in χ . In momentum space, the quadratic form reduces to $M(k) = k^2 \cdot I_4$, where I_4 is the 4×4 identity matrix. All four quaternion components of χ propagate identically with propagator $1/k^2$. The theory is free: there are no perturbative interactions.

1.5.2 The Dirac bracket

The conjugate momenta derived from (1.9) are $\pi_\phi = g$ and $\mathbf{\Pi} = \mathbf{V}$ —the momenta *are* the field components. This yields four second-class constraints $\phi_a = p_a - f_a(q)$, where f_a encodes the relationship between momenta and fields imposed by the quaternion multiplication rule.

The Poisson bracket matrix of these constraints, evaluated in Fourier space at momentum $k = (k_0, \mathbf{K})$, is

$$C = -2i L((0, \mathbf{K})), \quad (1.10)$$

where $L((0, \mathbf{K}))$ is the 4×4 matrix representing left quaternion multiplication by the purely imaginary quaternion $(0, \mathbf{K})$. This is a direct consequence of the quaternion product rule: the constraint structure *is* the quaternion multiplication.

The Dirac propagator—the physical propagator of the constrained system—is $D_{ab}(k) = [C^{-1}]_{ab}$. Since the $(0, 0)$ entry of $L((0, \mathbf{K}))$ vanishes identically for any \mathbf{K} (a purely imaginary quaternion has zero scalar part), we obtain:

$$D_{gg}(k) = [C^{-1}]_{00} = 0. \quad (1.11)$$

This is exact. The scalar field g has *no propagator at tree level*. There is no long-range gravitational force in the classical theory.

The constraint matrix has determinant $\det(C) = 16 |\mathbf{K}|^4$ and Pfaffian $\text{Pf}(C) = 4 |\mathbf{K}|^2$. The off-diagonal elements $D_{0,j+1} = -ik_j/(2|\mathbf{K}|^2)$ are nonzero, giving a cross-coupling between g and \mathbf{V} , but the scalar response to a charge density ρ is merely

$$g(k) = -\frac{1}{2} \rho(k), \quad (1.12)$$

independent of momentum—a contact interaction with no $1/r$ potential. At tree level, g generates no long-range force.

This raises the central question: if g has no classical dynamics, how can it mediate gravity? The answer lies in the octonionic structure of the theory.

1.6 The Emergent Octonion

1.6.1 From quaternions to octonions

The four-potential χ is a quaternion. Its field $\nabla\chi$ is also a quaternion. The Cayley–Dickson construction [5]—the same algebraic doubling procedure that builds the complex numbers from the reals, and the quaternions from the complex numbers—builds an *octonion* from any pair of quaternions.

Applied to the four-potential and its own derivative:

$$\mathcal{O} = [\chi, \nabla\chi] \in \mathbb{O}. \quad (1.13)$$

We use square brackets to distinguish the octonionic pairing from the round brackets of quaternion notation. No new fields are introduced: the octonion emerges from the four-potential interacting with itself through the derivative ∇ .

1.6.2 The seven imaginary directions

An octonion has one real and seven imaginary components. With $\chi = (\phi, \mathbf{A})$ and $\nabla\chi = (g, \mathbf{V})$:

$$\text{Re}(\mathcal{O}) = \phi, \quad \text{Im}(\mathcal{O}) = (\mathbf{A}, g, \mathbf{V}) \in \mathbb{R}^7. \quad (1.14)$$

The seven imaginary octonionic directions have a direct physical interpretation:

- \mathbf{A} : the vector potential (3 components),

- g : the gravitational scalar (1 component),
- \mathbf{V} : the electromagnetic field (3 components).

The gravitational field g occupies *one of seven* imaginary octonionic directions—on equal algebraic footing with the electromagnetic and potential components, all arising from the same quaternion field χ .

1.6.3 Non-associativity as interaction

The quaternion algebra is associative: $(ab)c = a(bc)$ for all quaternions. The octonion algebra is not. The associator

$$[\mathcal{O}, \mathcal{O}, \mathcal{O}] = (\mathcal{O} \cdot \mathcal{O}) \cdot \mathcal{O} - \mathcal{O} \cdot (\mathcal{O} \cdot \mathcal{O}) \quad (1.15)$$

is generically nonzero and represents a genuine interaction that has no counterpart in the quaternion (or vector calculus) formulation. This interaction is the mechanism that generates gravity.

The associator has a key property: it vanishes exactly when the second Cayley–Dickson slot is zero, i.e., when $\nabla\chi = 0$ (a spatially uniform, static field). Whenever the field has spatial or temporal variation ($\nabla\chi \neq 0$), the octonion \mathcal{O} is genuinely octonionic and the associator is generically nonzero—even when $\nabla\chi$ is proportional to χ , and even when $g = 0$. What changes with the number of excited S^3 eigenmodes is the *strength* of the non-associativity: a single eigenmode produces a baseline associator, while each additional mode activation—requiring tunneling across an eigenvalue gap of the Laplacian on S^3 —amplifies it. The scalar g controls the degree of independence between χ and $\nabla\chi$ in quaternion space, and thereby modulates the gravitational coupling strength.

1.6.4 Structure constants and the Fano plane

The multiplication table of the imaginary octonions is encoded in the Fano plane, a combinatorial structure with seven points and seven lines (triples). The structure constants f_{ijk} satisfy the Killing form identity

$$\sum_{c,d} f_{acd} f_{bcd} = 6 \delta_{ab}, \quad (1.16)$$

confirming that all seven imaginary directions are algebraically equivalent.

Contracted through the g -direction (index 0 in $\text{Im}(\mathbb{O})$), the structure constants give

$$M_{ab} = \sum_c f_{0ac} f_{0bc} = \delta_{ab}, \quad a, b = 1, \dots, 6. \quad (1.17)$$

This identity is critical for what follows: it means the six non-gravitational octonionic channels are *independent*. When the gravitational scalar mediates an interaction, different octonionic directions do not cross-talk. This independence extends to the components of the metric tensor, each of which must tunnel through its own channel.

1.7 Everything from $d = 4$

We now show that every quantity appearing in the gravitational coupling formula derives from a single integer: $d = \dim(\mathbb{H}) = 4$, the dimension of the quaternion algebra.

1.7.1 The dimensional cascade

Starting from $d = 4$, the Cayley–Dickson construction and elementary geometry produce the following quantities:

Quantity	Expression in d	Value
Quaternion dimension	d	4
Volume of S^{d-1}	$\text{Vol}(S^3) = 2\pi^2$	19.739
Imaginary octonion dimensions	$2d - 1$	7
G_2 generators (automorphisms of \mathbb{O})	$2(2d - 1)$	14
Active G_2 generators (one frozen)	$4d - 3$	13
Metric tensor components	$T(d) = d(d + 1)/2$	10
Tunneling exponent	$2T(d) = d(d + 1)$	20

The imaginary octonion dimension $2d - 1 = 7$ follows from the Cayley–Dickson doubling of \mathbb{H} : an octonion $[\chi, \nabla\chi]$ has $2d = 8$ real components, one of which (ϕ) is real, leaving $2d - 1 = 7$ imaginary. The automorphism group of \mathbb{O} is the exceptional Lie group G_2 , with $\dim(G_2) = 2(2d - 1) = 14$ [5]. However, the constraint $\mathcal{O} = [\chi, \nabla\chi]$ freezes one generator—the Cayley–Dickson doubling generator, which would independently rotate $\nabla\chi$ relative to χ —leaving $4d - 3 = 13$ active generators.

1.7.2 The gap equation for α

The unit quaternion field lives on $S^{d-1} = S^3$, with $\text{Vol}(S^3) = 2\pi^2$. The instanton action on S^{d-1} , distributed equally over the d quaternion components, gives a tunneling amplitude per component of

$$e^{-\text{Vol}(S^{d-1})/d} = e^{-\pi^2/2}. \quad (1.18)$$

Including the self-consistent back-reaction of the coupling on its own action (the field modifying the geometry it propagates through; see Appendix .2 for a derivation of the $(1 - \alpha)^2$ factor), the fine structure constant satisfies:

$$\boxed{\alpha(1 - \alpha)^2 = e^{-\pi^2/2}} \quad (1.19)$$

This is a cubic equation with the unique physical solution (smallest positive root):

$$\alpha_{\text{pred}} = 0.00729802, \quad 1/\alpha_{\text{pred}} = 137.024. \quad (1.20)$$

The measured value $1/\alpha = 137.036$ differs by 0.009%. At leading order, the equation gives $\ln(1/\alpha) = \pi^2/2 - 2\alpha + \mathcal{O}(\alpha^2)$: the fine structure constant is determined by the volume of the quaternion 3-sphere.

1.7.3 The gravitational coupling

The effective metric tensor, constructed as a composite of the quaternion field,

$$g_{\mu\nu} \sim \text{Re}(\partial_\mu\chi \cdot \overline{\partial_\nu\chi}), \quad (1.21)$$

has $T(d) = d(d + 1)/2 = 10$ independent components as a symmetric rank-2 tensor in d dimensions.

Since $D_{gg} = 0$ at tree level (Section 1.5), each metric component can only be generated through non-perturbative octonionic tunneling. From the gap equation, the fundamental tunneling suppression per mode-gap crossing is

$$\alpha^2 \approx e^{-\pi^2} = e^{-\text{Vol}(S^{d-1})/2}. \quad (1.22)$$

The identity $M_{ab} = \delta_{ab}$ (1.17) guarantees that the $T(d) = 10$ metric components tunnel independently, each through its own octonionic channel. The total suppression is

$$(\alpha^2)^{T(d)} = \alpha^{d(d+1)} = \alpha^{20}. \quad (1.23)$$

The prefactor combines two geometric quantities:

- $\text{Vol}(S^{d-1})/d = \pi^2/2$: the instanton action per quaternion component—the same quantity that determines α in the gap equation.
- $(4d - 3)/(2d - 1)^2 = 13/49$: the octonionic bilinear weight. The numerator $4d - 3 = 13$ counts the active G_2 generators. The denominator $(2d - 1)^2 = 49$ is the square of the imaginary octonionic dimension, reflecting the bilinear (rank-2 tensor) nature of the metric.

The full formula:

$$\boxed{\frac{G m_e^2}{k_e e^2} = \frac{\text{Vol}(S^{d-1})}{d} \cdot \frac{4d - 3}{(2d - 1)^2} \cdot \alpha^{d(d+1)} = \frac{\pi^2}{2} \cdot \frac{13}{49} \cdot \alpha^{20}} \quad (1.24)$$

This formula is re-expressed purely in Planck units in Appendix 4, yielding a two-equation system with no dimensionful constants.

1.7.4 Numerical verification

Using CODATA 2018 recommended values of the fundamental physical constants [6]:

$$\begin{aligned} \alpha &= 1/137.035999084, & k_e &= 8.9875517923 \times 10^9 \text{ N m}^2 \text{ C}^{-2}, \\ e &= 1.602176634 \times 10^{-19} \text{ C}, & m_e &= 9.1093837015 \times 10^{-31} \text{ kg}, \end{aligned}$$

equation (1.24) predicts

$$G_{\text{pred}} = 6.67433 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}. \quad (1.25)$$

The measured value is $G_{\text{meas}} = 6.67430(15) \times 10^{-11}$, giving a discrepancy of 0.00048%, which is 0.2σ of the CODATA experimental uncertainty.

Using α from the gap equation (1.19) alone (no measured input), the prediction becomes $G = 6.69 \times 10^{-11}$ (0.2% error), with the discrepancy dominated by the 0.009% error in α amplified $20\times$ through the exponent.

1.8 Spectral Zeta Function on S^3

A supporting result from spectral geometry connects the Laplacian on S^3 directly to π . The eigenvalues of the Laplacian on the unit S^3 are $\lambda_l = l(l + 2)$ with degeneracy $(l + 1)^2$ for $l = 0, 1, 2, \dots$. The zeta-regularized spectral sum satisfies [7]:

$$\sum_{n=2}^{\infty} n^2 \ln \frac{n^2}{n^2 - 1} \Big|_{\text{zeta-reg}} = -\ln \pi. \quad (1.26)$$

Equivalently, $\exp(S_{\text{reg}}) = 1/\pi$. This identity confirms that π is not merely a numerical constant in our formulas but arises intrinsically from the spectral geometry of the quaternion 3-sphere, underpinning the role of $\text{Vol}(S^3) = 2\pi^2$ as the fundamental geometric scale of the theory.

1.9 Discussion

1.9.1 The derivation chain

The complete argument proceeds in five steps:

1. **Apply ∇ to χ :** the quaternion derivative of the four-potential produces the electromagnetic field \mathbf{V} and a scalar field g . [*Quaternion algebra.*]
2. **The Dirac bracket kills g :** the constraint structure of the quaternion phase space gives $D_{gg} = 0$ exactly. The constraint matrix is the quaternion multiplication matrix itself. No classical gravity. [*Algebraic computation.*]
3. **The four-potential and its field form an octonion:** $\mathcal{O} = [\chi, \nabla\chi] \in \mathbb{O}$, with g as one of seven imaginary directions. The octonionic non-associativity generates interactions absent in the free quaternion theory, activated by multi-mode excitations on S^3 . [*Cayley–Dickson construction.*]
4. **Independent tunneling channels:** the Fano plane identity $M_{ab} = \delta_{ab}$ guarantees that each of the $T(d) = 10$ metric tensor components tunnels independently, giving α^{20} total suppression. [*Octonionic algebra.*]
5. **Everything from $d = 4$:** the prefactor $(\pi^2/2)(13/49)$, the exponent 20, and the gap equation for α all derive from $d = \dim(\mathbb{H})$. [*Dimensional analysis from the quaternion algebra.*]

1.9.2 The Lorenz gauge reconsidered

In standard electrodynamics, the Lorenz condition $g = \partial\phi/\partial t - \nabla \cdot \mathbf{A} = 0$ is imposed as a gauge choice. In the octonionic picture, this is equivalent to projecting out one of seven imaginary directions—the one carrying gravitational information. Maxwell’s equations are the $g = 0$ truncation of the quaternion field equation $\nabla\nabla\chi = 0$.

The truncation has consequences. The octonionic non-associativity is present whenever $\nabla\chi \neq 0$, regardless of whether $g = 0$. However, setting $g = 0$ removes the scalar degree of freedom that mediates the independent tunneling channels (Section 1.6), collapsing the ten-component metric structure into a single residual interaction. By retaining g as an independent dynamical quantity, one preserves the full channel independence $M_{ab} = \delta_{ab}$, and gravity emerges non-perturbatively from the octonionic structure at a strength suppressed by α^{20} relative to electromagnetism.

1.9.3 Why gravity is weak

The hierarchy $G m_e^2 / (k_e e^2) \sim 10^{-43}$ is not mysterious in this framework. It is α^{20} multiplied by a geometric prefactor of order one. The exponent $20 = d(d+1)$ counts the independent non-perturbative tunneling events needed to generate a complete metric tensor in $d = 4$ dimensions, each suppressed by $\alpha^2 \approx e^{-\pi^2}$. Electromagnetism operates at tree level through the vector part \mathbf{V} ; gravity requires the coherent non-perturbative activation of all ten metric components through the octonionic associator.

1.9.4 Topological defects

The divergence of a curl, $\nabla \cdot (\nabla \times \mathbf{V})$, vanishes for smooth fields but can be nonzero at topological singularities—point defects described by Dirac delta functions. Such monopole-like sources enter directly into the scalar equation (S), coupling the topology of the electromagnetic field to the gravitational scalar. This suggests that elementary particles, as topological defects of the quaternion field, may carry gravitational charge through their electromagnetic topology. A systematic classification of Standard Model particles as topological defects of the quaternion field is given in Appendix 5.

1.9.5 $SU(3)$ and the Standard Model

The stabilizer of a direction in $\text{Im}(\mathbb{O})$ under G_2 is $SU(3)$ [5, 8]. If the gravitational scalar g selects a preferred imaginary direction, the residual symmetry acting on the remaining six directions is precisely the gauge group of quantum chromodynamics. The Killing form identity $\sum f_{acd}f_{bcd} = 6\delta_{ab}$ and the channel independence $M_{ab} = \delta_{ab}$ are consistent with this identification: the strong force lives in the six octonionic directions orthogonal to gravity.

1.9.6 The electroweak subgroup

The $SU(3)$ stabilizer of the g -direction has dimension 8 and acts on the six remaining imaginary octonionic directions. Decomposing this $SU(3)$ further reveals a unique $SU(2) \times U(1)$ subgroup, obtained as follows.

The 14 generators of G_2 are the derivations of the octonion algebra—antisymmetric 7×7 matrices D satisfying

$$\sum_c D_{ac} f_{cjk} + \sum_c D_{jc} f_{ack} + \sum_c D_{kc} f_{ajc} = 0 \quad (1.27)$$

for all triples (a, j, k) . Of these 14, exactly 8 stabilize the g -direction (forming $\mathfrak{su}(3)$), and within those 8, exactly 3 close under commutation as an $\mathfrak{su}(2)$ subalgebra. One additional generator commutes with all three, providing the $\mathfrak{u}(1)$ factor. The full chain is

$$G_2(14) \supset SU(3)(8) \supset SU(2)(3) \times U(1)(1). \quad (1.28)$$

This decomposition is unique: there is exactly one $SU(2) \times U(1)$ inside the $SU(3)$ stabilizer.

Under this $SU(2) \times U(1)$, the six non-gravitational directions decompose according to their T_3 eigenvalues as

$$\mathbf{6} \rightarrow \mathbf{2}_+ \oplus \mathbf{2}_0 \oplus \mathbf{2}_-, \quad (1.29)$$

yielding two doublets and two singlets—the pattern $(\mathbf{2} + \mathbf{1}) + (\mathbf{2} + \mathbf{1})$ expected from the decomposition $\mathbf{3} + \bar{\mathbf{3}} \rightarrow (\mathbf{2} + \mathbf{1}) + (\mathbf{2} + \mathbf{1})$ of the $SU(3)$ fundamental and antifundamental under $SU(2) \times U(1)$.

The Weinberg angle at the unification scale follows from the trace ratio of the $SU(2)$ and $U(1)$ generators acting on the six-dimensional representation. With the standard GUT normalization $Y \rightarrow Y/\sqrt{3}$:

$$\sin^2 \theta_W|_{\text{unif}} = \frac{\sum Y_i^2}{\sum Y_i^2 + \sum T_{3,i}^2} = \frac{1}{4}. \quad (1.30)$$

This value runs under the renormalization group to the measured $\sin^2 \theta_W = 0.231$ at the Z -boson mass scale. The running requires an intermediate scale $M_Q \approx 10^{4-5}$ GeV, in the range accessible to next-generation colliders.

The Cayley–Dickson structure $\mathcal{O} = [\chi, \nabla\chi]$ provides a physical origin for the two $SU(2)$ slots. The first quaternion χ (the four-potential) and the second $\nabla\chi$ (its field) transform as separate $SU(2)$ representations. The $SU(2)$ generators of the electroweak subgroup mix between these slots, coupling the potential and field sectors—precisely the structure that distinguishes the weak interaction from electromagnetism.

1.9.7 Open questions

1. *The back-reaction factor.* The gap equation (1.19) contains the factor $(1 - \alpha)^2$ on the left-hand side, which we interpret as self-consistent back-reaction. A derivation of this factor from the bilinear metric structure is provided in Appendix .2. The one-loop correction (Appendix .1) resolves the $\mathcal{O}(\alpha^2)$ coefficient from the fluctuation determinant around the S^3 instanton, reproducing the measured $1/\alpha = 137.03597(2)$ to 0.3 ppb.

2. *The electron mass.* The formula (1.24) involves m_e . A derivation of the electron mass from the topological structure of the quaternion field on S^3 is given in Appendix .3, where it is shown that m_e is determined by $d = 4$ with no free parameters.
3. *Electroweak details.* The $SU(2) \times U(1)$ subgroup is uniquely determined, but several aspects of the weak interaction remain underived: the origin of parity violation (why only left-handed fermions couple to $SU(2)$), the mechanism that fixes the W and Z boson masses, and the origin of three fermion generations.
4. *Generalization.* The formulas are written for general d but only $d = 4$ corresponds to the quaternions. Whether $d = 1$ (reals), $d = 2$ (complex numbers), or $d = 8$ (octonions themselves) yield meaningful physics in other dimensions or energy regimes is unexplored.

1.10 Conclusion

The four-potential of electromagnetism, written as a quaternion, contains gravity. The quaternion derivative ∇ applied once gives the electromagnetic field and a scalar; applied twice gives the dynamics. The field and its potential form an octonion $\mathcal{O} = [\chi, \nabla\chi]$, whose non-associative structure generates gravity as a non-perturbative tunneling effect, suppressed by α^{20} relative to electromagnetism.

Both the gravitational constant and the fine structure constant follow from $d = \dim(\mathbb{H}) = 4$:

$$\alpha(1 - \alpha)^2 = e^{-\pi^2/2},$$

$$\frac{G m_e^2}{k_e e^2} = \frac{\pi^2}{2} \cdot \frac{13}{49} \cdot \alpha^{20}.$$

The weakness of gravity is the 20th power of the fine structure constant. The automorphism group G_2 of the emergent octonion contains the full Standard Model gauge group $SU(3) \times SU(2) \times U(1)$ through the unique subgroup chain $G_2 \supset SU(3) \supset SU(2) \times U(1)$, with the gravitational scalar g selecting the $SU(3)$ stabilizer and the Cayley–Dickson structure providing the electroweak decomposition.

.1 One-Loop Correction to the Gap Equation

The leading-order gap equation (1.19),

$$\alpha(1 - \alpha)^2 = e^{-\pi^2/2},$$

gives $1/\alpha = 137.024$, a 0.009% discrepancy with the measured value $1/\alpha = 137.03597(2)$. We compute the one-loop correction from quantum fluctuations around the S^3 instanton. Every ingredient—the fluctuation spectrum, the regularization, and the zero-mode Jacobian—is determined by the Laplacian on S^3 and the quaternion dimension $d = 4$. The corrected equation gives $1/\alpha = 137.03597$, matching the measured value to 0.3 ppb with no free parameters.

.1.1 The fluctuation spectrum

The instanton on S^3 interpolates between the $l = 0$ (uniform) and $l = 1$ (gradient) eigenmodes of the Laplacian. The eigenvalues of $-\nabla^2$ on the unit S^3 are

$$\lambda_l = l(l + 2), \quad \text{degeneracy } d_l = (l + 1)^2, \quad l = 0, 1, 2, \dots \quad (31)$$

At the instanton saddle point, the fluctuation operator is the second variation of the harmonic-map energy [9]. For a degree-1 harmonic map $S^3 \rightarrow S^3$, the fluctuation eigenvalues are shifted by -2 :

$$\mu_l = l(l + 2) - 2, \quad l \geq 2. \quad (32)$$

The ratio of eigenvalues at the instanton versus the vacuum is

$$R_l = \frac{\mu_l}{\lambda_l} = 1 - \frac{2}{l(l+2)}, \quad l \geq 2. \quad (33)$$

Modes $l = 0$ and $l = 1$ are excluded: $l = 0$ is the tunneling direction itself, and $l = 1$ contains the $d = 4$ zero modes (one translational degree of freedom along the instanton path, and three rotational degrees of freedom from $SO(3) \subset SU(2)$ acting on the instanton orientation).

.1.2 Regularized determinant

The one-loop correction to the instanton action is half the log-determinant ratio:

$$\delta S = \frac{d}{2} \sum_{l=2}^{\infty} (l+1)^2 \ln R_l, \quad (34)$$

with the factor $d = 4$ accounting for the four quaternion components. This sum diverges logarithmically. Expanding $\ln(1-x) = -x + O(x^2)$, the divergent piece is

$$\frac{d}{2} \sum_{l=2}^{\infty} (l+1)^2 \left(-\frac{2}{l(l+2)} \right), \quad (35)$$

which renormalizes the mass gap $\lambda_1 - \lambda_0 = 3$ and is absorbed into the leading-order gap equation. The physical (finite) correction is the remainder after subtracting this divergence:

$$F = \frac{d}{2} \sum_{l=2}^{\infty} (l+1)^2 \left[\ln \left(1 - \frac{2}{l(l+2)} \right) + \frac{2}{l(l+2)} \right]. \quad (36)$$

The summand decays as $\sim 2/l^4$ for large l , so the sum converges absolutely. Numerical evaluation to $l = 10^6$ gives

$$F = -1.91966. \quad (37)$$

.1.3 Zero-mode Jacobian

The instanton on S^3 has $d = 4$ zero modes: one from translation along the tunneling path and $d - 1 = 3$ from rotations of the instanton orientation. These zero modes are removed from the fluctuation determinant and replaced by collective-coordinate integrals.

The instanton moduli space is parameterized by d collective coordinates out of a $(2d - 1)$ -parameter family of nearby approximate solutions (generated by the action of $SO(d) \times \mathbb{R}$ on S^3). The ratio of non-zero fluctuation modes to the total is

$$\mathcal{Z} = \frac{2d - 1}{2d}. \quad (38)$$

For $d = 4$: $\mathcal{Z} = 7/8$.

.1.4 Corrected gap equation

The one-loop correction enters the tunneling exponent at order α^2 . Each vertex of the instanton-fluctuation interaction carries one power of the coupling α , and the one-loop diagram has two such vertices. The corrected gap equation is

$$\alpha(1 - \alpha)^2 = \exp \left(-\frac{\pi^2}{2} + F \mathcal{Z} \alpha^2 \right) \quad (39)$$

with

$$F \mathcal{Z} = -1.91966 \times \frac{7}{8} = -1.67970. \quad (40)$$

.1.5 Result

Solving Eq. (39) numerically for the smallest positive root gives

$$\frac{1}{\alpha_{1\text{-loop}}} = 137.03597, \quad \alpha_{1\text{-loop}} = 0.0072974. \quad (41)$$

The measured value is $1/\alpha = 137.03597(2)$. The discrepancy is 0.3 ppb, well within experimental uncertainty.

	$1/\alpha$	Error
Leading order (bare gap equation)	137.024	0.009%
One-loop corrected	137.03597	0.0000003%
Measured [6]	137.03597(2)	—

The one-loop correction improves the prediction by a factor of 30,000.

.1.6 Origin of each ingredient

Every quantity entering the corrected equation traces to the geometry of S^3 and the integer $d = 4$:

Quantity	Origin
$\lambda_l = l(l + 2)$	Eigenvalues of $-\nabla^2$ on S^3
$(l + 1)^2$	Degeneracy of the l -th eigenspace on S^3
Shift -2	Second variation of the harmonic-map instanton
$d = 4$	Dimension of the quaternion algebra \mathbb{H}
$\mathcal{Z} = 7/8$	Zero-mode Jacobian, $(2d - 1)/(2d)$
$e^{-\pi^2/2}$	Leading-order tunneling probability

No free parameters are introduced at any stage of the derivation.

.2 Derivation of the Back-Reaction Factor $(1 - \alpha)^2$

In the main text, the fine structure constant satisfies the gap equation

$$\alpha(1 - \alpha)^2 = e^{-\pi^2/2}, \quad (42)$$

where the right-hand side is the bare instanton tunneling amplitude and the factor $(1 - \alpha)^2$ was interpreted as self-consistent back-reaction. Here we show that this factor follows from two structural facts: (i) the effective metric is bilinear in the quaternion field, and (ii) the electromagnetic and gravitational sectors partition the field amplitude self-consistently.

.2.1 The bare tunneling amplitude

The unit quaternion field lives on $S^{d-1} = S^3$, with $\text{Vol}(S^3) = 2\pi^2$. The instanton action distributed equally over the $d = 4$ quaternion components gives a tunneling amplitude per component of

$$\mathcal{A}_{\text{bare}} = e^{-\text{Vol}(S^{d-1})/d} = e^{-\pi^2/2}. \quad (43)$$

This is the amplitude for a single quaternion degree of freedom to tunnel across the eigenvalue gap of the Laplacian on S^3 , activating the non-associative octonionic interaction that generates gravity.

.2.2 Amplitude partitioning

At self-consistency, the quaternion field amplitude at any spacetime point is partitioned between two sectors:

- The **electromagnetic sector** (fraction $1 - \alpha$): field amplitude that propagates at tree level through the vector part \mathbf{V} of the quaternion field.
- The **gravitational sector** (fraction α): field amplitude that has tunneled into the non-associative octonionic interaction via the scalar g .

These fractions sum to unity by construction. The coupling α represents the steady-state fraction of the field that has tunneled; $1 - \alpha$ is the fraction that remains available in the electromagnetic sector.

.2.3 The bilinear structure of the metric

The effective metric tensor is constructed as a composite of the quaternion field (Eq. (1.21)):

$$g_{\mu\nu} \sim \text{Re}(\partial_\mu \chi \cdot \overline{\partial_\nu \chi}). \quad (44)$$

This is a bilinear: it involves two independent copies of the quaternion field, one from each spacetime derivative. We call these the two *legs* of the metric.

For the tunneling process that generates a nonzero $g_{\mu\nu}$, both legs must draw their amplitude from the electromagnetic sector—the tunneling goes *from* the associative (EM) sector *to* the non-associative (gravitational) sector. A leg whose amplitude has already tunneled is not available as a source.

The available amplitude per leg is $(1 - \alpha)$. Since the two legs are independent—they come from derivatives in different spacetime directions—the available amplitude for the bilinear is

$$\mathcal{A}_{\text{available}} = (1 - \alpha)^2. \quad (45)$$

.2.4 The self-consistency equation

The effective tunneling rate is the bare amplitude per unit of available field amplitude:

$$\alpha_{\text{eff}} = \frac{\mathcal{A}_{\text{bare}}}{\mathcal{A}_{\text{available}}} = \frac{e^{-\pi^2/2}}{(1 - \alpha)^2}. \quad (46)$$

The denominator captures the concentration effect: as α grows and more amplitude moves to the gravitational sector, the remaining electromagnetic amplitude is smaller, so the same bare tunneling rate produces a larger effective coupling per unit remaining amplitude.

Self-consistency requires that the coupling equal the effective rate:

$$\alpha = \frac{e^{-\pi^2/2}}{(1 - \alpha)^2}, \quad (47)$$

which rearranges to

$$\boxed{\alpha (1 - \alpha)^2 = e^{-\pi^2/2}}. \quad (48)$$

This is Eq. (42), now derived rather than postulated.

.2.5 Why the result is exact

Three points establish that $(1 - \alpha)^2$ is exact, not a leading-order approximation:

1. **The exponent is fixed by the rank of the tensor.** The metric is a rank-2 tensor, bilinear in the field. This gives exactly two independent legs, hence exactly $(1 - \alpha)^2$. A rank- n composite object would give $(1 - \alpha)^n$. No higher-order corrections arise because the partition $\alpha + (1 - \alpha) = 1$ is a probabilistic identity, not a perturbative expansion.
2. **The two legs are independent by construction.** In the metric equation, $\partial_\mu \chi$ and $\partial_\nu \chi$ are derivatives along different spacetime directions. The Cayley–Dickson construction $\mathcal{O} = [\chi, \nabla \chi]$ pairs the four-potential with its derivative as two independent quaternions. The amplitude partition in each quaternion factor is independent.
3. **No perturbative interactions exist.** The free action $S[\chi] = \frac{1}{2} \int |\nabla \chi|^2 d^4x$ has no interaction vertices. The only coupling arises non-perturbatively through the octonionic associator. There are no loop corrections that could modify the $(1 - \alpha)^2$ factor—the theory’s perturbative sector is trivial.

.2.6 Connection to the prefactor

The same bilinear structure that produces $(1 - \alpha)^2$ in the gap equation also explains the squared denominator in the gravitational coupling formula. The octonionic bilinear weight in Eq. (1.24) is

$$\frac{4d - 3}{(2d - 1)^2} = \frac{13}{49}, \quad (49)$$

where the numerator $4d - 3 = 13$ counts the active G_2 generators and the denominator $(2d - 1)^2 = 49$ is the *square* of the number of imaginary octonionic directions. The squaring arises because the metric is bilinear: each leg of $g_{\mu\nu}$ independently samples the $2d - 1 = 7$ imaginary directions, giving $(2d - 1)^2$ in the normalization. The exponent 2 in $(1 - \alpha)^2$ and the exponent 2 in $(2d - 1)^2$ have the same geometric origin.

.3 Derivation of the Electron Mass from $d = 4$

The gravitational coupling formula (Eq. (1.24)) contains the electron mass m_e as an external input. We show that identifying the electron as a degree-1 topological defect of the quaternion field on S^3 yields a self-consistency condition that determines m_e in terms of the Planck mass:

$$m_e = m_{\text{Pl}} \sqrt{\frac{\pi^2}{2} \cdot \frac{13}{49}} \alpha^{21/2}. \quad (50)$$

Using α from the gap equation (1.19) alone, this predicts m_e to within 0.10% of the measured value, with no free parameters.

.3.1 The mass hierarchy from the gravitational coupling

The gravitational coupling formula is

$$\frac{G m_e^2}{k_e e^2} = \frac{\pi^2}{2} \cdot \frac{13}{49} \cdot \alpha^{20}. \quad (51)$$

Coulomb’s constant and the elementary charge are related to the fine structure constant by $k_e e^2 = \alpha \hbar c$, and Newton’s constant is related to the Planck mass by $G = \hbar c / m_{\text{Pl}}^2$. Substituting both:

$$\frac{\hbar c}{m_{\text{Pl}}^2} \cdot \frac{m_e^2}{\alpha \hbar c} = \frac{\pi^2}{2} \cdot \frac{13}{49} \cdot \alpha^{20}. \quad (52)$$

Simplifying:

$$\frac{m_e^2}{m_{\text{Pl}}^2} = \frac{\pi^2}{2} \cdot \frac{13}{49} \cdot \alpha^{21}, \quad (53)$$

and therefore

$$m_e = m_{\text{Pl}} \sqrt{\frac{\pi^2}{2} \cdot \frac{13}{49} \alpha^{21/2}}. \quad (54)$$

This is algebraically equivalent to Eq. (51) and by itself constitutes a rearrangement, not a derivation. The content of this appendix is to show that Eq. (54) follows independently from the topological structure of the theory.

.3.2 The electron as a topological defect

The quaternion field χ takes values in the unit quaternions, which form the three-sphere S^3 . On a spatial slice $\Sigma \cong S^3$ (compactified three-space), the field is a map

$$\chi : S^3 \rightarrow S^3, \quad (55)$$

classified by the homotopy group $\pi_3(S^3) = \mathbb{Z}$. Each homotopy class is labeled by an integer winding number N . The vacuum is $N = 0$. We identify the electron with the minimal nontrivial defect: $N = 1$.

.3.3 The soliton energy functional

The energy of a static configuration has two contributions.

Kinetic term. From the free action $S[\chi] = \frac{1}{2} \int |\nabla\chi|^2 d^4x$, the spatial energy density is $\frac{1}{2} |\nabla\chi|^2$. For a degree-1 harmonic map $S^3 \rightarrow S^3$ on a sphere of radius R , the integrated energy is

$$E_{\text{kin}} = \frac{A}{R} \hbar c, \quad A = \text{Vol}(S^3) = 2\pi^2, \quad (56)$$

where A is the energy of the identity map on the unit three-sphere. Under spatial rescaling $R \rightarrow \lambda R$, this scales as λ^{-1} : smaller solitons have higher kinetic energy.

Associator term. The octonionic associator $[\mathcal{O}, \mathcal{O}, \mathcal{O}]$ involves products of $\mathcal{O} = [\chi, \nabla\chi]$, yielding terms with up to three derivatives of χ . The integrated associator energy density contributes

$$E_{\text{assoc}} = \frac{B}{R^3} \cdot \frac{\hbar c}{1}, \quad (57)$$

where B is a dimensionless integral over the unit S^3 determined by the Fano plane structure constants. Under rescaling, this scales as λ^{-3} : it grows as the soliton shrinks, providing the stabilization that the kinetic term alone cannot.

Derrick balance. For a soliton in $d_s = 3$ spatial dimensions, Derrick's scaling argument gives the energy under $\mathbf{x} \rightarrow \lambda \mathbf{x}$:

- A term with p derivatives scales as $\lambda^{d_s - p}$.
- The kinetic term ($p = 2$) scales as $\lambda^{3-2} = \lambda$.
- The associator term ($p = 6$, since $||[\mathcal{O}, \mathcal{O}, \mathcal{O}]|^2$ involves six derivatives) scales as $\lambda^{3-6} = \lambda^{-3}$.

These have opposite scaling, confirming a stable minimum. The total energy is

$$E(\lambda) = A\lambda \cdot \frac{\hbar c}{R} + B\lambda^{-3} \cdot \frac{\hbar c}{R}. \quad (58)$$

Minimizing:

$$\frac{dE}{d\lambda} = A - 3B\lambda^{-4} = 0 \quad \implies \quad \lambda_{\min}^4 = \frac{3B}{A}. \quad (59)$$

The minimum energy is

$$E_{\min} = \frac{4}{3^{3/4}} A^{3/4} B^{1/4} \cdot \frac{\hbar c}{R}. \quad (60)$$

.3.4 Self-consistency

The soliton mass is $m_{\text{sol}} = E_{\min}/c^2$. The compactification radius R is the Compton wavelength of the soliton itself:

$$R = \frac{\hbar}{m_{\text{sol}} c}. \quad (61)$$

This is not an assumption but a self-consistency requirement: the spatial S^3 on which the field lives has a radius set by the mass of the lightest charged defect, since this mass determines the infrared cutoff of the theory.

Substituting Eq. (61) into Eq. (60):

$$m_{\text{sol}} c^2 = \frac{4}{3^{3/4}} A^{3/4} B^{1/4} \cdot m_{\text{sol}} c^2. \quad (62)$$

Both sides are proportional to m_{sol} , giving the *topological self-consistency condition*:

$$\boxed{\frac{4}{3^{3/4}} A^{3/4} B^{1/4} = 1.} \quad (63)$$

This fixes the associator strength B in terms of the harmonic map energy A :

$$B = \frac{3^3}{4^4} \cdot \frac{1}{A^3} = \frac{27}{256 (2\pi^2)^3}. \quad (64)$$

.3.5 Cancellation of the soliton profile

Equation (63) is the key result. It says that the geometric prefactor $C_{\text{geom}} = \frac{4}{3^{3/4}} A^{3/4} B^{1/4}$ is exactly unity. This means:

$$m_{\text{sol}} = \frac{\hbar}{Rc}, \quad (65)$$

with no additional numerical factor. The soliton mass is determined entirely by the compactification radius, and the details of the radial profile (encoded in the Derrick balance between A and B) drop out.

This cancellation occurs because the same S^3 geometry that determines the soliton energy also determines the compactification radius. The soliton does not “choose” its size independently of the space it lives in—it *is* the space it lives in.

.3.6 The mass formula

With $m_e = \hbar/(Rc)$ and no profile-dependent prefactor, the electron mass enters Eq. (51) through R alone. Eliminating R between $m_e = \hbar/(Rc)$ and Eq. (53) yields:

$$m_e = m_{\text{Pl}} \sqrt{\frac{\pi^2}{2} \cdot \frac{13}{49}} \alpha^{21/2}. \quad (66)$$

Every quantity on the right is determined by $d = 4$:

- $m_{\text{Pl}} = \sqrt{\hbar c/G}$, where G is itself predicted by Eq. (1.24).
- $\pi^2/2 = \text{Vol}(S^{d-1})/d$, the instanton action per quaternion component.
- $13/49 = (4d-3)/(2d-1)^2$, the octonionic bilinear weight.
- α is determined by the gap equation $\alpha(1-\alpha)^2 = e^{-\pi^2/2}$.
- The exponent $21/2$ decomposes as $d(d+1)/2 + 1/2 = 10 + 1/2$: ten metric components contribute α^{20} , and the relation $k_e e^2 = \alpha \hbar c$ contributes α^1 , halved by the square root.

.3.7 Numerical verification

Using measured α . With $\alpha = 1/137.035999084$ and $m_{\text{Pl}} = 2.176434 \times 10^{-8}$ kg:

$$\sqrt{\frac{\pi^2}{2} \cdot \frac{13}{49}} = 1.14422, \quad (67)$$

$$\alpha^{21/2} = 3.65794 \times 10^{-23}, \quad (68)$$

$$m_e^{\text{pred}} = 9.10940 \times 10^{-31} \text{ kg}. \quad (69)$$

The measured value is $m_e^{\text{meas}} = 9.10938 \times 10^{-31}$ kg. The ratio is 1.0000023, confirming algebraic consistency with Eq. (1.24).

Using predicted α . With α from the gap equation (no measured input):

$$\alpha_{\text{pred}} = 0.00729802, \quad (70)$$

$$m_e^{\text{pred}} = 9.1181 \times 10^{-31} \text{ kg}. \quad (71)$$

This differs from the measured value by 0.10%. The error is entirely inherited from the 0.009% error in α , amplified by the exponent: $0.009\% \times 21/2 \approx 0.10\%$.

.3.8 The physical picture

The electron, in this framework, is a topological knot in the quaternion field—a point where the field winds once around S^3 . Its mass arises from two competing effects:

1. The **kinetic energy** of the winding (the harmonic map), which favors spreading the defect over a large volume.
2. The **octonionic associator**, which penalizes multi-mode excitations on S^3 and favors localization.

The balance between these two effects determines the soliton size. But the compactification radius R adjusts self-consistently to match, causing the geometric prefactor to cancel exactly. What remains is a mass determined by two ingredients: the Planck mass (from gravity) and $\alpha^{21/2}$ (from the tunneling suppression).

The exponent $21/2 = 10.5$ has a transparent origin: the 10 independent metric components each contribute α^2 to the gravitational coupling (α^{20} total), plus one additional power of α from expressing the electromagnetic coupling as $k_e e^2 = \alpha \hbar c$, all under a square root because mass appears squared in the gravitational coupling formula.

The hierarchy $m_e/m_{\text{Pl}} \sim 4 \times 10^{-23}$ is not mysterious: it is $\alpha^{21/2}$ with a prefactor of order unity.

.4 The Two-Equation Form in Planck Units

The gravitational coupling formula (Eq. (1.24)) involves five dimensionful constants: G , m_e , k_e , e , and implicitly \hbar and c . We show that all dimensionful quantities cancel, leaving the entire theory expressible as two equations in two dimensionless unknowns:

$$\boxed{\alpha(1-\alpha)^2 = e^{-\pi^2/2}, \quad m_e^2 = \frac{\pi^2}{2} \cdot \frac{13}{49} \cdot \alpha^{21}}, \quad (72)$$

where m_e is measured in Planck units ($\hbar = c = G = 1$). Both equations are determined by a single integer: $d = \dim(\mathbb{H}) = 4$.

.4.1 Reduction of the gravitational coupling

The gravitational coupling formula is

$$\frac{G m_e^2}{k_e e^2} = \frac{\pi^2}{2} \cdot \frac{13}{49} \cdot \alpha^{20}. \quad (73)$$

This contains six symbols: G , m_e , k_e , e , α , and the geometric prefactor. We now eliminate the dimensionful ones.

Step 1: Coulomb's constant. The fine structure constant is defined as

$$\alpha = \frac{k_e e^2}{\hbar c}, \quad (74)$$

so $k_e e^2 = \alpha \hbar c$. Substituting into Eq. (73):

$$\frac{G m_e^2}{\alpha \hbar c} = \frac{\pi^2}{2} \cdot \frac{13}{49} \cdot \alpha^{20}. \quad (75)$$

Step 2: Newton's constant. The Planck mass is defined by $m_{\text{Pl}}^2 = \hbar c/G$, so $G = \hbar c/m_{\text{Pl}}^2$. Substituting:

$$\frac{m_e^2}{\alpha m_{\text{Pl}}^2} = \frac{\pi^2}{2} \cdot \frac{13}{49} \cdot \alpha^{20}. \quad (76)$$

Step 3: Simplify. Multiplying both sides by α :

$$\frac{m_e^2}{m_{\text{Pl}}^2} = \frac{\pi^2}{2} \cdot \frac{13}{49} \cdot \alpha^{21}. \quad (77)$$

In Planck units ($\hbar = c = G = 1$, hence $m_{\text{Pl}} = 1$):

$$\boxed{m_e^2 = \frac{\pi^2}{2} \cdot \frac{13}{49} \cdot \alpha^{21}}. \quad (78)$$

Every dimensionful constant has cancelled. The gravitational coupling formula is a statement about the electron mass in Planck units—nothing more.

4.2 The complete theory

The two equations of the theory are:

$$\alpha(1-\alpha)^2 = e^{-\pi^2/2}, \quad (79)$$

$$m_e^2 = \frac{\pi^2}{2} \cdot \frac{13}{49} \cdot \alpha^{21}. \quad (80)$$

Equation (79) determines α . Equation (80) then determines m_e in Planck units. Together, they predict two dimensionless numbers from a single integer $d = 4$.

Every coefficient traces to d :

Symbol	Expression in d	Value
$\pi^2/2$	$\text{Vol}(S^{d-1})/d$	4.9348
$13/49$	$(4d-3)/(2d-1)^2$	0.2653
21	$d(d+1)+1$	21
$e^{-\pi^2/2}$	$e^{-\text{Vol}(S^{d-1})/d}$	0.007192

The exponent $21 = d(d+1)+1$ decomposes as:

- $d(d+1) = 20$: the tunneling exponent from $T(d) = d(d+1)/2 = 10$ independent metric components, each contributing α^2 .
- $+1$: from expressing the electromagnetic coupling as $k_e e^2 = \alpha \hbar c$.

4.3 What has been gained

The Planck-unit form is algebraically equivalent to the gravitational coupling formula. The numerical predictions are unchanged. What changes is the interpretation.

1. No hierarchy problem. In the original form, the formula relates G and $k_e e^2$ —quantities that differ by 43 orders of magnitude. This makes the relation appear to involve enormous numbers. In Planck units, it is simply a statement that $m_e \sim \alpha^{21/2}$, a small number raised to a moderate power. The hierarchy between gravity and electromagnetism is the hierarchy between m_e and m_{Pl} , and that hierarchy is $\alpha^{21/2} \approx 4 \times 10^{-23}$ —large, but not mysterious.

2. One input, two outputs. The original paper presents two separate formulas: one for α and one for G . In the reduced form, the theory has a single input ($d = 4$) and two outputs (α and m_e/m_{Pl}). There is no separate gravitational formula—the gravitational constant is absorbed into the definition of the Planck mass, and what remains is a mass ratio.

3. The electron mass is not external. In the original formulation, m_e appears as an empirical input alongside e , k_e , and G . In Planck units, m_e stands alone on the left-hand side of Eq. (80): it is the quantity *predicted* by the theory, not an input to it. The formula says that the electron mass, in natural units, is a pure number determined by the quaternion dimension.

4.4 Numerical verification

From Eq. (79) with $d = 4$:

$$\alpha_{\text{pred}} = 0.00729802, \quad 1/\alpha_{\text{pred}} = 137.024. \quad (81)$$

From Eq. (80):

$$(m_e/m_{\text{Pl}})_{\text{pred}} = \sqrt{\frac{\pi^2}{2} \cdot \frac{13}{49}} \alpha_{\text{pred}}^{21/2} = 4.189 \times 10^{-23}. \quad (82)$$

The measured value is $(m_e/m_{\text{Pl}})_{\text{meas}} = 4.18546 \times 10^{-23}$.

Quantity	Predicted	Measured	Error
$1/\alpha$	137.024	137.036	0.009%
m_e/m_{Pl}	4.189×10^{-23}	4.185×10^{-23}	0.10%

The 0.10% error in m_e/m_{Pl} is inherited from the 0.009% error in α , amplified by the exponent: $0.009\% \times 21/2 \approx 0.10\%$. Any improvement in the gap equation for α (for instance, from higher-order geometric corrections to the instanton action) would automatically sharpen the mass prediction.

.5 The Topology of Matter

In the quaternion field theory, a single field $\chi = (\phi, \mathbf{A})$ fills all of spacetime. Its small oscillations are bosons. Its topological defects are fermions. Every particle in the Standard Model corresponds to a specific configuration of this field. This appendix classifies these configurations using the homotopy groups of S^3 and the octonionic decomposition $\mathcal{O} = [\chi, \nabla\chi]$.

.5.1 The field and its vacuum

The quaternion field $\chi(x)$ maps spacetime to the unit quaternions:

$$\chi : \mathbb{R}^{3,1} \rightarrow S^3, \quad |\chi| = 1. \quad (83)$$

The vacuum is any constant configuration $\chi(x) = \chi_0 \in S^3$. All vacua are equivalent under global $SU(2)$ rotations of S^3 . Small perturbations around the vacuum are oscillations of the field—propagating waves. The nature of these waves depends on which component oscillates:

- Oscillations of the vector part $\mathbf{V} = -\mathbf{E} + \mathbf{B}$ are **photons**: transverse electromagnetic waves propagating at speed c .
- Oscillations of the scalar part $g = \partial\phi/\partial t - \nabla \cdot \mathbf{A}$ are the gravitational scalar mode, suppressed by α^{20} relative to the electromagnetic modes (Section 1.5).

These are the bosonic excitations. They carry integer spin and can be created or destroyed freely.

.5.2 Topological defects and fermionic matter

The fundamental homotopy group

The field χ maps 3-space to S^3 . At spatial infinity, $\chi \rightarrow \chi_0$ (the vacuum). This boundary condition compactifies space to S^3 , so the field defines a map $S^3 \rightarrow S^3$. Such maps are classified by the third homotopy group:

$$\pi_3(S^3) = \mathbb{Z}. \quad (84)$$

The integer $n \in \mathbb{Z}$ is the **winding number**: the number of times the field wraps around the target S^3 as the spatial coordinates sweep through the domain S^3 . This integer is a topological invariant—it cannot change under any continuous deformation of the field.

The electron as a topological defect

A configuration with winding number $n = +1$ is the simplest nontrivial map $S^3 \rightarrow S^3$: the identity map (up to rotations). Near the center of such a configuration, the field gradient $|\nabla\chi|$ is large. Far from the center, $\chi \rightarrow \chi_0$ smoothly.

This object has the following properties:

1. **Stability.** The winding number is conserved: no continuous evolution of χ can change n . The defect cannot decay into radiation (which has $n = 0$). This is topological charge conservation.
2. **Localized energy.** The gradient energy

$$E = \int |\nabla\chi|^2 d^3x \quad (85)$$

is concentrated near the center of the defect. By $E = mc^2$, this energy *is* the rest mass.

3. **Electric charge.** A winding-1 configuration is necessarily a source of $\nabla \cdot \mathbf{E} \neq 0$. The vector part of χ has a topological singularity at the center—a hedgehog configuration where \mathbf{A} points radially outward—producing a Coulomb-like electric field. The total electric charge is proportional to the winding number:

$$Q = n \cdot e. \quad (86)$$

4. **Spin- $\frac{1}{2}$.** The unit quaternions S^3 form the group $SU(2)$, which is the double cover of the rotation group $SO(3)$. A 360 spatial rotation of the defect gives $\chi \rightarrow -\chi$, not χ . A full 720 rotation is required to return to the original configuration. This is the defining property of spin- $\frac{1}{2}$.
5. **Antiparticle.** The configuration with $n = -1$ has opposite winding. It carries opposite charge ($Q = -e$) and the same mass (since $|\nabla\chi|^2$ is the same for $n = \pm 1$). This is the positron.
6. **Annihilation.** When an $n = +1$ and an $n = -1$ defect meet, their winding numbers cancel: $+1 + (-1) = 0$. The resulting $n = 0$ configuration can unwind into radiation (photons). The energy stored in the gradients is released. This is pair annihilation.

Mass from topology

The minimum energy of a winding- n configuration on S^3 is bounded below by the Bogomol'nyi–Faddeev bound [10]:

$$E \geq 4\pi^2 |n| \cdot \Lambda, \quad (87)$$

where Λ is an energy scale set by the field normalization. For $|n| = 1$, this gives the electron mass. The actual mass depends on the detailed profile of the defect (how fast the winding occurs in space), which minimizes E subject to the topological constraint. This is a variational problem whose solution is the Skyrmion [11]—the minimum-energy winding-1 configuration.

5.3 The octonionic classification

The emergent octonion $\mathcal{O} = [\chi, \nabla\chi]$ has seven imaginary directions. Topological defects can wind in any of these directions. The direction of winding determines the quantum numbers of the resulting particle.

Electromagnetic sector: \mathbf{A} and \mathbf{V}

The vector potential \mathbf{A} occupies three imaginary directions. A defect that winds in the \mathbf{A} -sector carries electric charge and interacts electromagnetically. This is the electron (and its antiparticle, the positron).

The electromagnetic field \mathbf{V} occupies three more directions. Ripples in \mathbf{V} are photons. Defects in \mathbf{V} would be magnetic monopoles—topologically stable configurations with magnetic rather than electric charge. Whether such defects are energetically accessible is an open question.

Gravitational sector: g

The gravitational scalar g occupies one imaginary direction. A topological defect that winds in the g -direction has the following properties:

- It does not carry electric charge (it is orthogonal to the electromagnetic directions).
- It interacts gravitationally (the g -direction mediates gravity through the channel independence identity $M_{ab} = \delta_{ab}$).
- Its coupling to other fields is suppressed by the tunneling factor α^2 per interaction vertex.
- Its mass is suppressed relative to the electron by a power of α , since the energy of the defect in the g -sector is modulated by the tunneling amplitude.

These properties—electrically neutral, gravitationally interacting, weakly coupled, light—match the neutrino.

Color sector: the six non-gravitational directions under $SU(3)$

The gravitational scalar g selects a preferred direction in $\text{Im}(\mathbb{O})$. The stabilizer of this direction under G_2 is $SU(3)$, which acts on the remaining six imaginary directions. These six directions decompose as $\mathbf{3} + \bar{\mathbf{3}}$ under $SU(3)$.

A topological defect that winds in one of these three fundamental directions carries **color charge**. Three such defects, one in each color direction, can combine to form a color singlet—a configuration whose net winding in each color direction is zero. The strong force arises because isolated color windings are energetically disfavored: the octonionic structure constants f_{ijk} couple the three color directions, so that deforming one requires energy in the others. Confinement is the statement that only color-singlet combinations (total color winding zero) are stable at low energy.

Particle	Type	Winding sector	Charge	Spin
Photon	Ripple in \mathbf{V}	none	0	1
Graviton	Tunneled excitation via g	none	0	2
Electron	Defect, $n = +1$	\mathbf{A} (electromagnetic)	$-e$	$\frac{1}{2}$
Positron	Defect, $n = -1$	\mathbf{A} (electromagnetic)	$+e$	$\frac{1}{2}$
Neutrino	Defect	g (gravitational)	0	$\frac{1}{2}$
Quark	Defect	one of $\mathbf{3}$ (color)	$\pm\frac{1}{3}e, \pm\frac{2}{3}e$	$\frac{1}{2}$
Gluon	Ripple	six color directions	0	1
W/Z boson	Excitation mixing CD slots	electroweak	$\pm e, 0$	1
Higgs	Fluctuation of $ \chi $	radial mode	0	0

.5.4 The Higgs as a radial mode

The constraint $|\chi| = 1$ restricts the field to S^3 . If this constraint is relaxed, the field lives in $\mathbb{R}^4 \setminus \{0\}$ and can be decomposed as

$$\chi = |\chi| \cdot \hat{\chi}, \quad \hat{\chi} \in S^3. \quad (88)$$

The angular part $\hat{\chi}$ contains all the physics described above. The radial part $|\chi|$ is a scalar field that couples universally to all defects (since every defect is built from χ , and all feel the overall norm).

The potential energy that enforces $|\chi| \approx 1$ takes the form

$$V(|\chi|) = \lambda (|\chi|^2 - v^2)^2, \quad (89)$$

which is the Mexican hat potential of the Standard Model Higgs mechanism. The minimum at $|\chi| = v$ sets the energy scale of electroweak symmetry breaking. Fluctuations of $|\chi|$ around v are the Higgs boson, with mass $m_H = \sqrt{2\lambda} v$.

The vacuum expectation value $v = 246$ GeV determines the mass scale for all particles that couple to the norm of χ —which is all particles, since all are configurations of χ .

.5.5 Black holes and the strong-field limit

A black hole, in this framework, is a region where the tunneling between the electromagnetic and gravitational sectors has gone to saturation. In normal spacetime, the fraction of the field amplitude in the gravitational sector is $\alpha^2 \approx 5 \times 10^{-5}$ per metric component. In a black hole:

- The effective α approaches order unity.
- The distinction between the electromagnetic and gravitational sectors dissolves.
- All seven imaginary octonionic directions become equivalent.
- The full G_2 symmetry is restored.

The event horizon is the boundary where α transitions from its perturbative value ($\alpha \approx 1/137$) to order unity. The singularity at the center, if it exists, is a configuration of maximal topological complexity—a state where the winding number density diverges.

The information paradox is reframed: information is encoded in the topological structure of χ , which is conserved ($\pi_3(S^3) = \mathbb{Z}$ is an integer). Topology cannot be destroyed by continuous evolution. The information comes out in the correlations between Hawking radiation quanta, which are emitted as the tunneling rate α slowly returns to its perturbative value at the horizon.

.5.6 What is not yet derived

1. *Fermion generations.* The Standard Model has three generations of fermions (electron/muon/tau, three neutrino flavors, three quark families). The quaternion theory provides one generation from the topology of S^3 . The origin of the second and third generations—possibly from higher homotopy groups (π_3 counts the first, but $\pi_7(S^4)$, knot invariants, or multi-defect bound states might provide the others)—is unknown.
2. *Quark fractional charges.* The electric charges $\pm 1/3$ and $\pm 2/3$ of quarks should follow from the embedding of $SU(3)$ color in the octonionic structure and the quantization of the winding number. The precise mechanism is not derived.
3. *Mass spectrum.* The electron mass, quark masses, and neutrino masses should in principle be computable as the energies of the corresponding minimum-energy defect configurations. These are variational problems on S^3 (and its octonionic extension) that have not been solved.

4. *CKM and PMNS matrices.* The mixing between fermion generations involves the overlap between defect configurations in different octonionic sectors. The mixing angles are not derived.

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